Theory of thin airfoils in fluids of high electrical conductivity

By W. R. SEARS AND E. L. RESLER
College of Engineering, Cornell University, Ithaca, N.Y.

(Received 21 July 1958)

Steady, plane flow of incompressible fluid past thin cylindrical obstacles is treated with two different orientations of the undisturbed, uniform magnetic field; namely, parallel and perpendicular, respectively, to the undisturbed, uniform stream. In the first case, the flow of an infinitely conducting fluid is shown to be irrotational and current-free except for surface currents at the walls of the obstacles. With large but finite conductivity the surface currents are replaced by thin boundary layers of large current density.

In the second case, for infinite conductivity the flow field is made up of an irrotational current-free part and a system of waves involving currents and vorticity extending out from the body. For large, finite conductivity these waves attenuate exponentially with distance from the body.

In both cases the forces on sinusoidal walls and on airfoils are calculated. In the second case positive drag occurs.

Introduction

In this paper we study the steady flow past thin cylindrical bodies of an incompressible fluid of high electrical conductivity. In particular, the approximation of infinite conductivity will be adopted for the most part. Although this approximation is not believed to be appropriate for most problems in the field called 'magneto-aerodynamics' (see, Resler & Sears 1958), it should be appropriate in other situations which involve greater 'magnetic Reynolds numbers'. Such situations may include flows involving high gas temperatures, or flows of liquid metals. Our results will also be of some interest in illustrating phenomena that occur when the fields induced by the motion are large, as contrasted with the typical aeronautical cases where these induced fields are often negligible.

We are concerned solely with the domain where the fluid may be considered a continuum and where the flow of electricity may be described by Ohm's Law. Other simplifications are made in writing the pertinent equations of motion in the next section.

The basic equations

Steady flow of an inviscid fluid of constant density ρ and conductivity σ is described by the following equations:

continuity, $$\operatorname{div} \mathbf{q} = 0; \qquad (1)$$

momentum,
$$\mathbf{q}\,.\,\nabla\mathbf{q}+\frac{1}{\rho}\nabla p=\frac{1}{\rho}\mathbf{j}\times\mathbf{H}; \tag{2}$$

Ohm's Law,
$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{q}\times\mathbf{H}); \tag{3}$$

and Maxwell's equations,
$$4\pi\mathbf{j}=\operatorname{curl}\mathbf{H}; \tag{4}$$
$$\operatorname{curl}\mathbf{E}=0; \tag{5}$$
$$\operatorname{div}\mathbf{H}=0; \tag{6}$$

where $\mathbf{q}$ denotes the fluid velocity, p the pressure, $\mathbf{j}$ the current density, $\mathbf{H}$ the magnetic field vector, and $\mathbf{E}$ the electric field vector. We are employing electromagnetic units, so that the permeability of empty space is 1, and we assume that the fluid has the same permeability. In equation (3) we neglect the current component that arises from the flow of charged fluid particles, because this term can be shown to be small, compared to the terms retained, in the type of flow considered here. In equation (4) we neglect 'displacement currents' for the same reason.

By substitution for $\mathbf{j}$ from equation (4) into equation (2), the momentum equation becomes
$$\mathbf{q}\,.\,\nabla\mathbf{q}+\frac{1}{\rho}\nabla p=\frac{1}{4\pi\rho}\{\mathbf{H}\,.\,\nabla\mathbf{H}-\tfrac{1}{2}\nabla H^2\}. \tag{7}$$

Similarly, both $\mathbf{j}$ and $\mathbf{E}$ can be eliminated from Ohm's Law by means of equations (4) and (5). The result, after some straight-forward vector calculus and use of equations (1), (5) and (6), is
$$-\operatorname{curl}(\mathbf{q}\times\mathbf{H})=\mathbf{q}\,.\,\nabla\mathbf{H}-\mathbf{H}\,.\,\nabla\mathbf{q}=\frac{1}{4\pi\sigma}\nabla^2\mathbf{H}. \tag{8}$$

The problem is now reduced to the simultaneous solution for given boundary conditions of equations (7) and (8) for $\mathbf{q}$, $\mathbf{H}$ and p, subject to the conditions $\operatorname{div}\mathbf{q}=0=\operatorname{div}\mathbf{H}$.

Small-perturbation flow: Case I

Consider now the case of plane, steady flow in which the stream velocity and the magnetic field are uniform and parallel except for small disturbances; i.e. we suppose
$$\mathbf{q}=(U+u,v,0),\quad\text{where}\quad u,v\ll U, \tag{9}$$
and
$$\mathbf{H}=(H_0+h_x,h_y,0),\quad\text{where}\quad h_x,h_y\ll H_0 \tag{10}$$

We assume now that the ratios of perturbations to free-stream values are of the same order for both the velocity field and the magnetic field, and this will be shown later to be correct, at least for fluids of high conductivity.

To first order, then, the equation of momentum, equation (7), becomes
$$U\frac{\partial\mathbf{v}}{\partial x}+\frac{1}{\rho}\nabla p=\frac{1}{4\pi\rho}\left\{H_0\frac{\partial\mathbf{h}}{\partial x}-H_0\nabla h_x\right\}, \tag{11}$$

where $\mathbf{v}$ denotes the vector (u,v) and $\mathbf{h}$ the vector (h_x,h_y). The x-component equation in equation (11) can be integrated immediately, yielding
$$\rho Uu+p=p_\infty, \tag{12}$$

where p_∞ denotes the undisturbed static pressure.

The y-component equation, upon substitution of $-\rho U(\partial u/\partial y)$ for $\partial p/\partial y$, becomes

$$\Omega = \frac{H_0}{4\pi\rho U}\xi, \tag{13}$$

where Ω and ξ are respectively the curls of the vectors $\mathbf{q}$ and $\mathbf{H}$; i.e.

$$\Omega \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{14}$$

$$\xi \equiv \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} = 4\pi j_z, \tag{15}$$

where j_z is the z-component of current density.

Similarly, equation (8) becomes

$$U\frac{\partial \mathbf{h}}{\partial x} - H_0\frac{\partial \mathbf{v}}{\partial x} = \frac{1}{4\pi\sigma}\nabla^2\mathbf{h}. \tag{16}$$

Infinite conductivity

For fluids of very large electrical conductivity, neglect the right-hand side of equation (16). This equation is then integrable and becomes

$$\frac{\mathbf{h}}{H_0} - \frac{\mathbf{v}}{U} = \text{function of } y$$
$$= 0, \tag{17}$$

where it is assumed that the function of y is evaluated at large y where the stream and magnetic field are undisturbed.

This relation states that the velocity and magnetic fields are distorted in exactly the same way; the streamlines and magnetic lines of force are always parallel. This is, of course, a result of the infinite-conductivity approximation. It is not, in fact, restricted to the small-perturbation case, as will now be shown. For $\sigma \to \infty$, equation (8) becomes

$$\text{curl}\,(\mathbf{q} \times \mathbf{H}) = 0. \tag{18}$$

But in plane flow, since $\mathbf{q} \times \mathbf{H}$ has only a z-component, equation (18) requires that

$$\mathbf{q} \times \mathbf{H} = \text{constant} \tag{19}$$

Thus if this vector product vanishes anywhere, as it does in the undisturbed part of the flow in the present problem, it must vanish everywhere, and the streamlines and magnetic lines must be parallel to one another.

Returning now to the small-perturbation case, we see, in view of equation (17), that the vorticity Ω is proportional to the current density, so that equation (13) becomes

$$\Omega = \frac{H_0^2}{4\pi\rho U^2}\Omega. \tag{20}$$

Thus, except for the particular case where the stream speed U is equal to the 'Alfvén velocity' $\mathbf{H}_0/\sqrt{(4\pi\rho)}$, we conclude that the flow is irrotational and that the current-density j_z is zero to first order. In the special case the disturbance produced by the body is propagated in the form of an Alfvén wave, relative to the undisturbed stream, at just the speed of the body. Hence, the body moves in a 'cavity' of its own shape, and exerts no force on the fluid.

At all other stream speeds the flow field about the body is just the same as in a non-conducting fluid, and is determined by the boundary conditions and the required cyclic constants (circulations). We therefore carry over to this field all the familiar solutions of plane, small-perturbation, irrotational, incompressible flow, such as are provided by thin-airfoil theory.

There are, however, important modifications to the force on the body surface, since there are, in general, *surface currents* at such an interface. These are sheets of infinite current density, which allow the solution formed above (equation (17)) to exist and the required boundary conditions at the body surface to be satisfied. We shall illustrate this effect by means of three examples.

(1) *Infinite sinusoidal wall (insulator)*

Let the contour of an insulating wall be defined by $y = \epsilon \cos \lambda x$ where $\epsilon \ll \lambda^{-1}$. For brevity we can adopt the familiar complex notation, real parts being implied everywhere, and write $y = \epsilon\, e^{i\lambda x}$. The first-order solution for the flow problem is well known:

$$v = iU\epsilon\lambda\, e^{i\lambda(x+iy)} = iu. \tag{21}$$

Thus, the perturbation of the magnetic field is given by

$$h_y = iH_0\epsilon\lambda\, e^{i\lambda(x+iy)} = ih_x. \tag{22}$$

The surface current, in the z-direction, is given by equation (4) in the form

$$4\pi J_s = -h_x(x, +0) + h_x(x, -0), \tag{23}$$

where J_s denotes the current per unit length, flowing in the z-direction, and the values assumed at the interface between fluid and wall have been replaced approximately by the values at $y = \pm 0$ in the usual way. We have determined $h_x(x, +0)$ in equation (22).

Within the insulator there can be no currents, so again the magnetic field must be curl-free. The boundary condition at the interface is provided by the requirement of continuous h_y; hence $h_y(x, -0)$ is given by equation (22). The disturbance must vanish as $y \to -\infty$. The solution, for $y < 0$, is

$$h_y = iH_0\epsilon\lambda\, e^{i\lambda(x-iy)} = -ih_x. \tag{24}$$

The surface current, according to equation (23), is therefore given by

$$4\pi J_s = -2H_0\epsilon\lambda\, e^{i\lambda x}. \tag{25}$$

According to equation (12), the pressure distribution throughout the fluid is the same as in the analogous flow of a non-conductor. The force exerted on the wall, however, is given by the difference between the local static pressure and the force, $H_0 J_s$ per unit area, arising from the surface current. Let this net pressure be denoted by $p_n(x)$; then, to first order,

$$\begin{aligned} p_n(x) &= p(x, +0) - H_0 J_s && (26)\\ &= p_\infty - \rho U u(x, +0) + (H_0^2/2\pi)\,\epsilon\lambda\, e^{i\lambda x} \\ &= p_\infty - \rho U^2 \epsilon\lambda\, e^{i\lambda x} + (H_0^2/2\pi)\,\epsilon\lambda\, e^{i\lambda x} \\ &= p_\infty - \rho U^2 \epsilon\lambda\, e^{i\lambda x}\{1 - 2m^{-2}\}, && (27) \end{aligned}$$

where m denotes the ratio of the stream speed U to the Alfvén velocity,

$$m \equiv \frac{U}{H_0/\sqrt{(4\pi\rho)}}.$$

Thus, the amplitude of the net pressure disturbance on the wall is reduced by the magneto-hydrodynamic effect as compared to the usual result. This pressure disturbance vanishes at a flow speed $\sqrt{2}$ times as great as the Alfvén velocity, and is opposite in sign for greater field strengths H_0.

In these results, the singular case mentioned above for a stream flowing at the Alfvén velocity does not appear, of course, since it involves rotational flow and electric-current flow in the fluid in the z-direction. Presumably, it could only be set up by means of carefully controlled initial conditions.

(2) *Lifting airfoil without thickness*

The classical irrotational small-perturbation flow past a body of this category may be represented by a vortex sheet which produces a discontinuous velocity component u and continuous v. According to equation (17), the discontinuity of u is proportional to the discontinuity of h_x at the same location on the airfoil; the latter determines the surface current as in equation (23). Thus the net force distribution (lift loading) on the airfoil is given by

$$l_n(x) = -\rho U\{u(x,-0)-u(x,+0)\}+(H_0/4\pi)\{h_x(x,-0)-h_x(x,+0)\} \quad (28)$$

$$= 2u(x,+0)\{\rho U - H_0^2/(4\pi U)\} = 2\rho U u(x,+0)\{1-m^{-2}\}. \quad (29)$$

The coefficients of lift and moment become, therefore,

$$c_l = c_{l_0}\{1-m^{-2}\} \quad \text{and} \quad c_m = c_{m_0}\{1-m^{-2}\}, \quad (30)$$

where c_{l_0} and c_{m_0} are the analogous coefficients for the same airfoil in a non-conducting fluid.

It is interesting to notice that, in the approximation of infinite conductivity, it is immaterial whether the airfoil itself is a conductor or insulator. In either case the current is the same, being determined solely by the discontinuity of h_x.

The total current flowing in the z-direction is proportional to the total circulation Γ:

$$(J_s)_{\text{total}} = \frac{H_0}{4\pi U}\Gamma. \quad (31)$$

(3) *Symmetrical airfoil with thickness (insulator)*

The flow about a body of this category is represented by a source-sink distribution, which produces discontinuous v and continuous u. The distortion of the magnetic field is such that the lines of force follow the streamlines around the body. Such a field can be constructed by imagining that magnetic sources and sinks are distributed along the axis. There is no net current at the body and the total force on the cylinder is zero.

However, to determine the pressure distribution on the surface one must give up the magnetic-source-sink approximation and consider the magnetic field within the body; again the resulting discontinuity of h_x involves a surface current.

But the magnetic field within the body is extremely simple, for the streamline pattern has been constructed to satisfy the condition of vanishing velocity component normal to the surface. Thus the magnetic field satisfies the same condition: vanishing normal component. But, if the body is an insulator, the magnetic field is harmonically distributed inside, and the field strength is zero. The 'convection' of magnetic lines by the flow therefore involves very large forces at the body surface, for the surface current is of zero-th, rather than first, order; namely, the surface current is, on the upper surface of the body

$$J_s = -\frac{1}{4\pi}(H_0 + h_x) = -\frac{H_0}{4\pi}\left(1 + \frac{u}{U}\right). \tag{32}$$

It will be seen that equation (32) predicts a zero-order surface current even for vanishing U. This is, of course, a consequence of the approximations of our theory, since we have essentially required the streamlines to be distorted around the body even at vanishing stream speed. It does seem qualitatively correct that, even at very small stream speeds in a fluid of great conductivity, the magnetic field must be parted by the body and that large forces must result. The force cannot be calculated by our approximate, first-order formula (26), in view of the large perturbation of the magnetic field; it is of order H_0^2.

Since the conclusion $\mathbf{q} \times \mathbf{H} = 0$ is not limited to small-perturbation flow (equation (19)), it seems clear that the vanishing of the interior field is not a consequence of the small-perturbation approximation but only of the assumptions of infinite conductivity and plane, steady flow.

Finite conductivity

If σ is large but finite, the right-hand side of equation (8) is small, and this equation may be attacked by a 'boundary-layer' approximation. We shall postpone a general treatment to a subsequent paper and consider here only the corresponding small-perturbation case, i.e. equation (16), which also takes on a boundary-layer character for large σ, but has the virtue of being linear.

We first operate on equation (16) with the curl and then substitute for Ω by means of equation (13). The result is

$$\nabla^2\xi - \kappa\frac{\partial \xi}{\partial x} = 0, \tag{33}$$

where

$$\kappa \equiv 4\pi\sigma U(1 - m^{-2}). \tag{34}$$

The usual boundary-layer arguments show that, as $\kappa \to \infty$, the appropriate approximation to equation (33) within a boundary layer of thickness $O(\kappa^{-\frac{1}{2}})$ is

$$\frac{\partial^2\xi}{\partial y^2} - \kappa\frac{\partial \xi}{\partial x} = 0. \tag{35}$$

More precisely, the approximation applies for large 'magnetic Reynolds numbers' $R_m = UL\sigma$, where L is the characteristic body dimension, and applies in a layer of thickness $O(LR_m^{-\frac{1}{2}})$, provided that m^2 is not close to 1. The analogy with the familiar viscous boundary layer is quite complete, except, of course, that linearization is usually not appropriate for such layers. At the outside edge of the magnetic boundary layer, equation (33) goes over into $\partial\xi/\partial x = 0$, which represents our irrotational, current-free flow field.

The magnetic boundary layer therefore serves to eliminate surface-current sheets, replacing them by boundary layers of large current density, just as a viscous boundary layer can be thought of as replacing a vortex sheet.

In the magnetic boundary layer the approximate expression for ξ is $-\partial h_x/\partial y$; thus equation (35) can be integrated once to yield

$$\frac{\partial^2 h_x}{\partial y^2} - \kappa \frac{\partial h_x}{\partial x} = F(x), \tag{36}$$

where $F(x)$ is recognized as the value of $-\kappa\, \partial h_x/\partial x$ at the edge of the layer, i.e. in the potential flow.

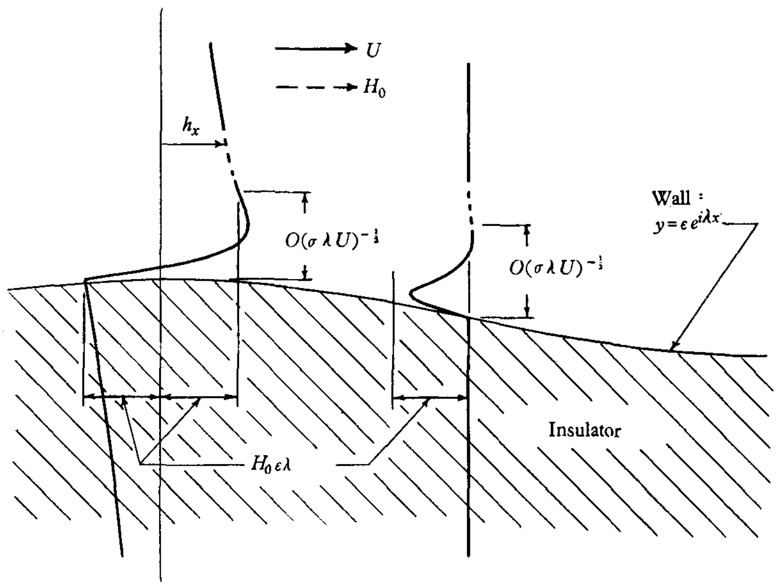

FIGURE 1. Distribution of the magnetic-field perturbation component h_x in the magnetic boundary layer of flow past a sinusoidal insulating wall. The undisturbed magnetic field is parallel to the undisturbed stream.

Let us apply these equations to the case of the infinite wavy wall. The boundary conditions for this problem are consistent with the small-perturbation assumption; namely,

(i) at the wall, from equation (24), $h_x = -H_0 \epsilon\lambda\, e^{i\lambda x}$,

(ii) at the outer edge of the magnetic boundary layer, from equation (22),

$$h_x = H_0 \epsilon\lambda\, e^{i\lambda x}.$$

The appropriate solution of equation (36) is

$$h_x(x,y) = H_0 \epsilon\lambda\, e^{i\lambda x}\left\{1 - 2\exp\left[\sqrt{\frac{\lambda\kappa}{2}}\,(-i-1)\,y\right]\right\}. \tag{37}$$

This solution is presented graphically in figure 1 for a case $m < 1$.

Small-perturbation flow: Case II

Consider now the case of plane, steady flow in which the stream velocity and the magnetic field are uniform and perpendicular except for small disturbances; i.e. let

$$\mathbf{q} = (U+u, v, 0), \quad \text{where} \quad u, v \ll U, \tag{38}$$

and

$$\mathbf{H} = (h_x, H_0+h_y, 0), \quad \text{where} \quad h_x, h_y \ll H_0. \tag{39}$$

It will again be assumed that the ratios of perturbation to free-stream values are of the same order for both velocity and magnetic fields.

To first order, the equation of momentum, equation (7), then becomes

$$U\frac{\partial \mathbf{v}}{\partial x}+\frac{1}{\rho}\nabla p = \frac{1}{4\pi\rho}\left\{H_0\frac{\partial \mathbf{h}}{\partial y}-H_0\nabla h_y\right\}. \tag{40}$$

Taking the divergence of both sides of this equation, we obtain

$$\nabla^2\left(p+\frac{H_0}{4\pi}h_y\right) = 0, \tag{41}$$

i.e. the quantity $p+(H_0/4\pi)h_y$ (which might be called the total pressure perturbation including magnetic pressure) is a harmonic function in the x, y-plane.

The form assumed by Ohm's Law, equation (8), in this case is

$$U\frac{\partial \mathbf{h}}{\partial x}-H_0\frac{\partial \mathbf{v}}{\partial y} = \frac{1}{4\pi\sigma}\nabla^2\mathbf{h}. \tag{42}$$

We can obtain interesting forms of equations (40) and (42) by employing the curl operator. Equation (40) becomes

$$U\frac{\partial \Omega}{\partial x} = \frac{H_0}{4\pi\rho}\frac{\partial \xi}{\partial y}, \tag{43}$$

while equation (42) becomes

$$U\frac{\partial \xi}{\partial x}-H_0\frac{\partial \Omega}{\partial y} = \frac{1}{4\pi\sigma}\nabla^2\xi. \tag{44}$$

By cross-differentiation these can be put into the forms

$$\frac{\partial^2\xi}{\partial x^2}-\frac{1}{m^2}\frac{\partial^2\xi}{\partial y^2} = \frac{1}{4\pi\sigma U}\nabla^2\frac{\partial \xi}{\partial x}, \tag{45}$$

and

$$\frac{\partial^2\Omega}{\partial x^2}-\frac{1}{m^2}\frac{\partial^2\Omega}{\partial y^2} = \frac{H_0}{(4\pi U)^2\rho\sigma}\nabla^2\frac{\partial \xi}{\partial y}. \tag{46}$$

Infinite conductivity

Before discussing this case in detail, it may be well to point out that the configuration of the undisturbed flow, having flow velocity and magnetic field at right angles, is a possible one even for fluids of infinite conductivity. To be sure, equation (8) with right-hand side equal to zero is often interpreted as stating that 'magnetic lines of force are convected with the fluid'. However, this is only a convenient interpretation of the fact that the number of lines enclosed by any closed contour is constant if the contour is convected with the fluid (see Cowling

(1957), p. 5, equations (1–13), and Hayes (1949)). In the present case the magnetic field must extend to $x = \pm\infty$; i.e. the magnet must be infinite in length. A possible realization of this configuration might involve a channel which closes on itself within a uniform magnetic field, provided that the radius of the channel were so large as to make the approximation of plane flow tenable.

For $\sigma \to \infty$, the right-hand sides of equations (45) and (46) are negligible and both Ω and ξ satisfy the wave equation. Their general solutions may then be written immediately; they must both be of the form $f(x-my)+g(x+my)$. Let us adopt the notation

$$\left.\begin{aligned} \Omega &= F_1'(x-my)+G_1'(x+my), \\ \xi &= F_2'(x-my)+G_2'(x+my). \end{aligned}\right\} \qquad (47)$$

These are equivalent to

$$\left.\begin{aligned} u &= \frac{m}{1+m^2}\{F_1(x-my)-G_1(x+my)\}+\frac{\partial\phi_1}{\partial x}, \\ v &= \frac{1}{1+m^2}\{F_1(x-my)+G_1(x+my)\}+\frac{\partial\phi_1}{\partial y}, \\ h_x &= \frac{m}{1+m^2}\{F_2(x-my)-G_2(x+my)\}+\frac{\partial\phi_2}{\partial x}, \\ h_y &= \frac{1}{1+m^2}\{F_2(x-my)+G_2(x+my)\}+\frac{\partial\phi_2}{\partial y}, \end{aligned}\right\} \qquad (48)$$

where the condition $\operatorname{div}\mathbf{v} = 0$ has also been satisfied, provided that

$$\nabla^2\phi_1 = 0 = \nabla^2\phi_2.$$

But upon substituting equations (47) into the momentum equations, equation (40), we are able to integrate with respect to x and y, respectively, obtaining

$$Uu+p/\rho = -(H_0/4\pi\rho)\{F_2(x-my)+G_2(x+my)\}+C, \qquad (49)$$

and

$$Uu+p/\rho = -(U/m)\{-F_1(x-my)+G_1(x+my)\}+C, \qquad (50)$$

where C is a constant. This requires that

$$\left.\begin{aligned} F_2(x-my) &= -\sqrt{(4\pi\rho)}\,F_1(x-my), \\ G_2(x+my) &= \sqrt{(4\pi\rho)}\,G_1(x+my). \end{aligned}\right\} \qquad (51)$$

The rotational parts of the velocity and magnetic fields are therefore related in a simple manner. Moreover, the irrotational parts are also related. First, we have equation (42) with right-hand side equal to zero. Substitution of equations (48) and (51) into this equation leads to

$$U\frac{\partial^2\phi_2}{\partial x^2} = H_0\frac{\partial^2\phi_1}{\partial x\,\partial y} \quad \text{and} \quad U\frac{\partial^2\phi_2}{\partial x\,\partial y} = H_0\frac{\partial^2\phi_1}{\partial y^2}, \qquad (52)$$

from which

$$U\frac{\partial\phi_2}{\partial x} = H_0\frac{\partial\phi_1}{\partial y}, \qquad (53)$$

except for a constant, which must be put equal to zero in view of the undisturbed-flow conditions.

Second, we can make use of equation (19), which applies to this case. For this perturbation case under consideration, it becomes, to first order,

$$H_0 u + U h_y = 0. \qquad (54)$$

Upon substitution for u and h_y from equations (48), and in view of equations (51), it follows that

$$U\frac{\partial \phi_2}{\partial y} = -H_0\frac{\partial \phi_1}{\partial x}. \tag{55}$$

In summary, equations (48) involve four unknown functions; $F_1(x-my)$, $G_1(x+my)$, $\phi_1(x,y)$, and $\phi_2(x,y)$ (since F_2 and G_2 are linearly related by (51)); and the last two of these are related by equations (53) and (55). We now illustrate the application of this theory by working out three examples.

(1) *Infinite sinusoidal wall* (*insulator*)

We again consider the flow over the wall whose contour is given by $y = \epsilon\, e^{i\lambda x}$. The boundary condition at the wall is, to first order

$$v(x,0) = UY'(x) = iU\lambda\epsilon\, e^{i\lambda x}. \tag{56}$$

As a second boundary condition we require that no perturbations are propagated toward the wall from the undisturbed stream; i.e. for this flow

$$G_1(x+my) = G_2(x+my) = 0. \tag{57}$$

Since all remaining perturbation quantities will be sinusoidal, we introduce the notation

$$F_1(x-my) = F_0\, e^{i\lambda(x-my)}, \tag{58}$$

$$\phi_1(x,y) = f_1\, e^{i\lambda(x+iy)} = -i\frac{U}{H_0}\phi_2(x,y), \tag{59}$$

where F_0 and f_1 are constants, and the last equality follows from equations (53) and (55).

No currents flow in the insulator, so that the description of the magnetic field for $y < 0$ is provided by another potential function, say $\phi_3(x,y)$, where, for $y < 0$,

$$h_x = \frac{\partial \phi_3}{\partial x} = i\lambda f_3\, e^{i\lambda(x-iy)} = i\frac{\partial \phi_3}{\partial y} = ih_y, \tag{60}$$

where f_3 is a constant.

As before, to satisfy equation (6) we require continuity of the component h_y at the wall-fluid interface. Thus

$$\lambda f_3 = -\frac{\sqrt{(4\pi\rho)}}{1+m^2}F_0 - i\frac{H_0}{U}\lambda f_1, \tag{61}$$

while equation (56) states that

$$\frac{1}{1+m^2}F_0 - \lambda f_1 = iU\lambda\epsilon. \tag{62}$$

In the present problem, moreover, we shall also require the component h_x to be continuous at the solid-fluid interface, because a first-order surface current would produce first-order force components in the tangential direction and there is no mechanism here to resist such force components. Thus,

$$i\lambda f_3 = -\frac{m\sqrt{(4\pi\rho)}}{1+m^2}F_0 - \frac{H_0}{U}\lambda f_1. \tag{63}$$

The simultaneous solution of equations (61), (62) and (63) leads immediately to the results

$$F_0 = -2U\lambda\epsilon \frac{1+m^2}{2i+m+im^2}, \tag{64}$$

and

$$f_1 = U\epsilon \frac{m^2-im}{2i+m+im^2}, \tag{65}$$

from which the wall pressure is found, using equation (50), to be

$$p(x,0) = p_\infty - \rho U^2 \lambda\epsilon \frac{m^2-2im-2}{m(m-2i)} e^{i\lambda x}. \tag{66}$$

The function of m that appears in equation (66) is plotted in figure 2 in the form of a vector diagram giving the amplitude and phase of the pressure perturbation.

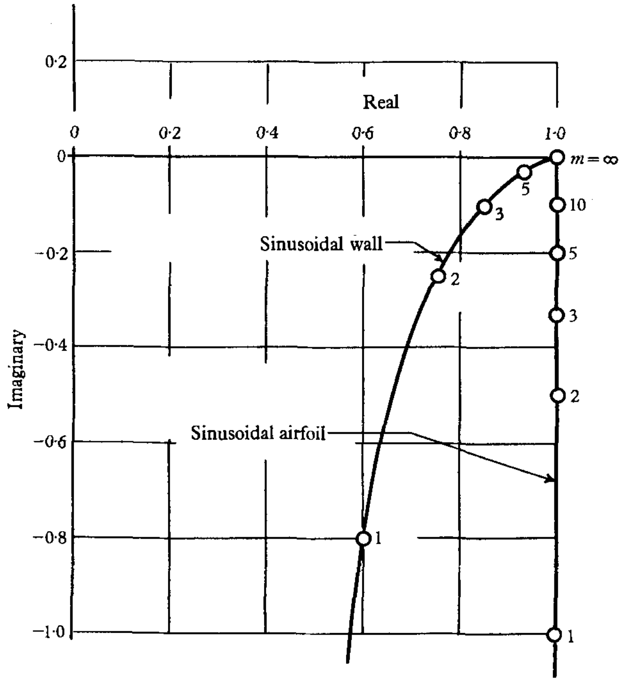

FIGURE 2. Vector diagram showing real and imaginary parts of the functions of m for (*a*) the pressure on a sinusoidal insulating wall as given by equation (66), and (*b*) the pressure (or lift) on a sinusoidal airfoil as given by equation (79). The undisturbed magnetic field is perpendicular to the undisturbed stream.

(2) *Lifting airfoil without thickness* (*insulator*)

Suppose the boundary condition is

$$v(x,0) = UY'(x) \quad \text{for} \quad -b \leqslant x \leqslant b, \tag{67}$$

so that $y = Y(x)$ gives the airfoil shape. This requires both

$$\frac{1}{1+m^2} F_1(x) + \left(\frac{\partial \phi_1}{\partial y}\right)_{y=+0} = UY',$$

and

$$\frac{1}{1+m^2} G_1(x) + \left(\frac{\partial \phi_1}{\partial y}\right)_{y=-0} = UY'. \tag{68}$$

Once again we require both h_x and h_y to be continuous at the airfoil, for the reasons set forth above. The first of these two conditions, in view of equations (48) and (51), means that

$$\frac{m\sqrt{(4\pi\rho)}}{1+m^2}\{F_1(x)-G_1(x)\} = \left(\frac{\partial\phi_2}{\partial x}\right)_{y=+0} - \left(\frac{\partial\phi_2}{\partial x}\right)_{y=-0} \tag{69}$$

or, with equations (68),

$$m\sqrt{(4\pi\rho)}\left\{\left(\frac{\partial\phi_1}{\partial y}\right)_{y=-0} - \left(\frac{\partial\phi_1}{\partial y}\right)_{y=+0}\right\} = \left(\frac{\partial\phi_2}{\partial x}\right)_{y=+0} - \left(\frac{\partial\phi_2}{\partial x}\right)_{y=-0}. \tag{70}$$

If this equation is compared with equation (53), with due regard for signs, it becomes clear that $\partial\phi_1/\partial y$ and $\partial\phi_2/\partial x$ must be continuous at the airfoil. Thus ϕ_1 may involve a vortex sheet and ϕ_2 a sheet of magnetic sources, at most. In view of this conclusion, we have from the boundary conditions (68) that

$$F_1(x) = G_1(x). \tag{71}$$

At this point it is easy to verify from equation (48) that $u(x, +0) = -u(x, -0)$; but equation (54) states that $u(x,y)$ is proportional to $h_y(x,y)$, which has been made continuous at the airfoil. The conclusion is therefore that

$$u(x, \pm 0) = 0 = h_y(x, 0). \tag{72}$$

We have succeeded in relating explicitly the rotational and irrotational parts of the flow; i.e.

$$\frac{1}{1+m^2}F_2(x) = -\left(\frac{\partial\phi_2}{\partial y}\right)_{y=+0} \tag{73}$$

or

$$F_1(x) = -\frac{1+m^2}{m}\left(\frac{\partial\phi_1}{\partial x}\right)_{y=+0}. \tag{74}$$

The boundary condition (68) can now be expressed in terms of the unknown potential function $\phi_1(x,y)$ alone:

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{y=+0} - m\left(\frac{\partial\phi_1}{\partial y}\right)_{y=0} = -mUY' \tag{75}$$

in the interval $-b \leqslant x \leqslant b$.

Boundary-value problems of this type were treated by Rott & Cheng (1954), and a procedure for constructing their solutions was given. Applied to the present problem, it leads to the result

$$\frac{\partial\phi_1}{\partial x} - i\frac{\partial\phi_1}{\partial y} = i\left\{\left(\frac{z-b}{z+b}\right)^\beta \frac{mU}{\pi\sqrt{(1+m^2)}}\int_{-b}^{b}\left(\frac{b+s}{b-s}\right)^\beta \frac{Y'(s)\,ds}{s-z}\right\}, \tag{76}$$

where $z = x+iy$ and $\beta = (\frac{1}{2})-(1/\pi)\tan^{-1}(1/m)$, the $\tan^{-1}$ being in the first quadrant. However, this solution does not seem particularly illuminating in the present studies, and we give instead the solutions for two cases where simpler results are obtained.

(*a*) Suppose, for example, that the airfoil is sinusoidal in shape ($Y = \epsilon\, e^{i\lambda x}$) and of infinite chord ($b = \infty$). All terms in equation (75) are then sinusoidal, and it is easily ascertained that

$$\phi_1(x,y) = -\frac{mi}{m+i}U\epsilon\, e^{i\lambda(x+iy)} \quad \text{for} \quad y > 0, \tag{77}$$

whence
$$F_1(x) = -\frac{1+m^2}{m+i} U\lambda\epsilon\, e^{i\lambda x} \tag{78}$$

and, from equation (50), the lift loading $l(x)$ is

$$l(x) = 2\rho U^2 \lambda\epsilon\left(1-\frac{i}{m}\right) e^{i\lambda x}. \tag{79}$$

The function of m that appears in equation (79) is plotted in figure 2.

(*b*) A family of solutions of equation (75) is given by the so-called Glauert series of thin-airfoil theory (Glauert (1926), p. 88); namely,

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{y=+0} = B_0 \frac{1-\cos\theta}{\sin\theta} + \sum_{n=1} B_n \sin n\theta, \tag{80}$$

$$\left(\frac{\partial\phi_1}{\partial y}\right)_{y=0} = -B_0 - \sum_{n=1} B_n \cos n\theta, \tag{81}$$

and their linear combination according to equation (75),

$$-mUY'(x) = B_0\left\{\frac{1-\cos\theta}{\sin\theta} + m\right\} + \sum_{n=1} B_n\{\sin n\theta + m\cos n\theta\}, \tag{82}$$

where $x/b = \cos\theta$.

The profiles of airfoils of this family are found by integration of equation (82). Their lift loading is, from equations (50) and (74),

$$\begin{aligned} l(x) &= -\frac{2}{m}\rho U F_1(x) \\ &= 2\rho\frac{1+m^2}{m^2} U\left\{B_0\frac{1-\cos\theta}{\sin\theta} + \sum_{n=1} B_n \sin n\theta\right\}. \end{aligned} \tag{83}$$

The terms in equation (80) have been selected to satisfy the Kutta–Joukowski condition of finite velocity at the trailing edge, as can easily be checked using equations (48), (71) and (74).

In figure 3 are shown the airfoils of this family obtained by using the terms in B_1 only, for several values of m. These airfoils all produce elliptical lift loading of magnitude proportional to $(1+m^2)/m^2$; i.e. their lift coefficients are

$$c_l = 2\pi\frac{1+m^2}{m^2}\frac{B_1}{U}. \tag{84}$$

It seems clear that this lift, for $m < \infty$, is associated with positive drag.

(3) *Symmetrical airfoil with thickness (insulator)*

Within the airfoil we have

$$\operatorname{div}\mathbf{h} = 0 = \operatorname{curl}\mathbf{h}. \tag{85}$$

Again, the boundary values for the inner field are given by the values of h_x and h_y assumed at the airfoil surface; there can be no surface currents for the reasons explained above. Thus the boundary values of h_x and h_y are of first order, and so are $\partial h_x/\partial x$ and $\partial h_y/\partial x$. It follows immediately from equations (85) that $\partial h_y/\partial y$ and

$\partial h_x/\partial y$ are of first order. By integration with respect to y, we have then that the values of both components at the top surface of the airfoil differ from the values at the lower surface by higher-order quantities.

We therefore take both h_x and h_y to be continuous across the singularity sheet that represents the airfoil in the limit, i.e.

$$h_x(x, +0) = h_x(x, -0),$$
$$h_y(x, +0) = h_y(x, -0). \tag{86}$$

In view of equation (54), u must also be continuous across the airfoil.

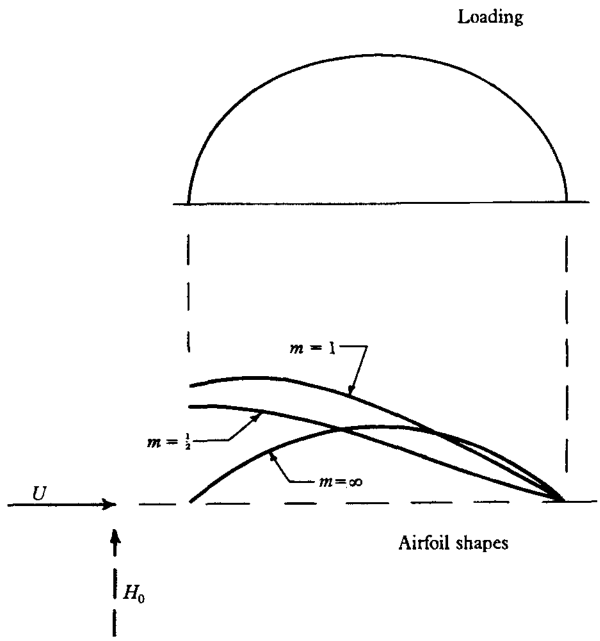

FIGURE 3. Lift loading and corresponding geometries of three airfoils producing elliptic loading for different values of m. The undisturbed magnetic field is perpendicular to the undisturbed stream.

With the aid of equations (48) and (51), it follows that

$$\frac{\sqrt{(4\pi\rho)}}{1+m^2}\{F_1(x)+G_1(x)\} = \left(\frac{\partial\phi_2}{\partial y}\right)_{y=+0} - \left(\frac{\partial\phi_2}{\partial y}\right)_{y=-0} \tag{87}$$

and, using also equation (53),

$$\frac{m\sqrt{(4\pi\rho)}}{1+m^2}\{F_1(x)-G_1(x)\} = \frac{H_0}{U}\left\{\left(\frac{\partial\phi_1}{\partial y}\right)_{y=+0} - \left(\frac{\partial\phi_1}{\partial y}\right)_{y=-0}\right\}. \tag{88}$$

The boundary conditions at the airfoil are

$$v(x, \pm 0) = \pm UY'(x) \quad \text{for} \quad -b \leqslant x \leqslant b, \tag{89}$$

so that, from equations (48),

$$\left(\frac{\partial\phi_1}{\partial y}\right)_{y=+0} - \left(\frac{\partial\phi_1}{\partial y}\right)_{y=-0} = 2UY'(x) - \frac{1}{1+m^2}\{F_1(x)-G_1(x)\}. \tag{90}$$

By comparison of equations (88) and (90), we have

$$F_1(x) - G_1(x) = 2UY'(x). \tag{91}$$

Now, the continuity of u, in view of equations (48), means that

$$\frac{m}{1+m^2}\{F_1(x)+G_1(x)\} = \left(\frac{\partial \phi_1}{\partial x}\right)_{y=-0} - \left(\frac{\partial \phi_1}{\partial x}\right)_{y=+0}, \tag{92}$$

while the boundary conditions (89) state that

$$-\frac{1}{1+m^2}\{F_1(x)+G_1(x)\} = \left(\frac{\partial \phi_1}{\partial y}\right)_{y=+0} + \left(\frac{\partial \phi_1}{\partial y}\right)_{y=-0}. \tag{93}$$

Suppose that the potential function ϕ_1 involves both a vortex sheet and a source-sink sheet at the airfoil. Then the right-hand side of equation (92) is the vortex-strength distribution, while the right-hand side of equation (93) is twice the corresponding vertical-velocity component, since the source strength does not contribute to either. These quantities are seen to be proportional to one another. But it can be shown (e.g. Rott & Cheng 1954) that the only vortex distributions that satisfy this homogeneous condition are singular at both $x = \pm b$, and this would necessarily lead to infinite loadings there in the present problem. Thus, by means of the Kutta–Joukowski condition at the trailing edge, we conclude that ϕ_1 arises from a source-sink sheet alone.

Then

$$F_1(x) = -G_1(x), \tag{94}$$

and equation (87) states that the magnetic-field singularity at the airfoil is a current sheet only. Combination of equations (88) and (91) yields the following simple formula for the distribution of the fluid sources and the strength of the fictitious current sheet:

$$\frac{H_0}{U}\left(\frac{\partial \phi_1}{\partial y}\right)_{y=+0} = \left(\frac{\partial \phi_2}{\partial x}\right)_{y=+0} = \frac{m\sqrt{(4\pi\rho)}}{1+m^2}F_1(x) = \frac{m\sqrt{(4\pi\rho)}}{1+m^2}UY'(x). \tag{95}$$

The corresponding values of $\partial\phi_1/\partial x$ at the sheet are given by

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{y=0} = \frac{1}{\pi}U\frac{m^2}{1+m^2}P\int_{-b}^{b}\frac{Y'(s)\,ds}{x-s}, \tag{96}$$

where the symbol P means that the Cauchy principal value is to be taken.

The surface pressure can now be calculated from equations (49) or (50):

$$p(x,+0) = p_\infty + \frac{1}{m(1+m^2)}\rho U^2 Y'(x) - \frac{1}{\pi}\rho U^2\frac{m^2}{1+m^2}P\int_{-b}^{b}\frac{Y'(s)\,ds}{x-s}. \tag{97}$$

The pressure on the lower surface, $p(x,-0)$, differs from p_∞ by an equal amount, and there is no lift.

Examples of pressure distributions calculated from equation (97) are easy to construct. We exhibit here in figure 4 the case of a thin elliptic cylinder. The pressure distribution is plotted for various values of m. The distortion of the distribution from its familiar symmetrical form as m is reduced from infinity obviously implies non-vanishing drag. Its value is

$$D = \frac{2}{m(1+m^2)}\rho U^2\int_{-b}^{b}\{Y'(x)\}^2\,dx. \tag{98}$$

Finite conductivity

For large but finite conductivity, we return to equations (45) and (46) and assume that the right-hand terms are small. The solution can then be carried out by introducing damping into the sinusoidal-wave solutions found in the preceding section, in the same way that viscous damping is introduced in problems of acoustics. This is the same process as Alfvén (1950) used to calculate the damping of propagating magneto-hydrodynamic waves.

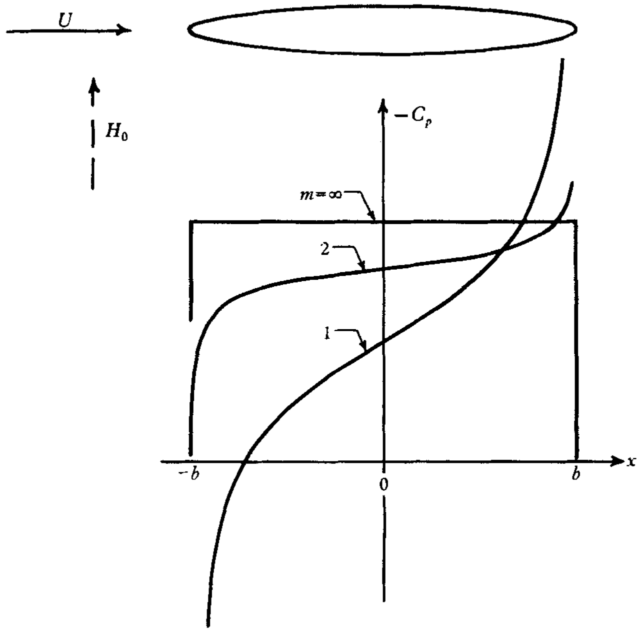

FIGURE 4. Pressure distribution on a thin elliptic cylinder at zero incidence for several values of m. The undisturbed magnetic field is perpendicular to the undisturbed stream.

Thus, for the infinite sinusoidal wall we write, in analogy with equations (47) and (58),

$$\xi = \kappa\, e^{i\lambda(x-my)}\, e^{-\mu y}, \tag{99}$$

where κ and μ are constants. Substituting this into equation (45), we have, after neglecting terms of order μ^2 in comparison with terms of order μ,

$$\mu = \frac{(1+m^2)\, m\lambda^2}{8\pi U\sigma - 2m^2\lambda} \approx \frac{(1+m^2)\, m\lambda^2}{8\pi U\sigma}. \tag{100}$$

To help interpret this result, let y_σ denote the value of y at which the waves are damped to $1/e$ times their amplitude at the wall; i.e.

$$y_\sigma = \mu^{-1} = \frac{8\pi U\sigma}{(1+m^2)\, m\lambda^2}. \tag{101}$$

This means that the time to damp to $1/e$, since the waves propagate in the y direction with the Alfvén speed, is

$$\frac{\sqrt{(4\pi\rho)}\, y_\sigma}{H_0} = \frac{2}{\pi}\frac{\sigma l_w^2}{m^2+1}, \tag{102}$$

where l_w denotes the wave length of the wall, $2\pi/\lambda$. The right-hand side of equation (102) is seen to be proportional to a diffusion time, namely, the time required for the magnetic field to diffuse over a distance equal to the minimum spacing between the waves, $l_w/\surd(m^2+1)$. In this interpretation our result agrees with Alfvén's in the investigation mentioned above [Alfvén (1950), p. 82].

The conclusion reached here is that one effect of small, non-vanishing resistance is damping of the vorticity and current in the waves that emanate from the solid surface. The pressure, including the values of pressure at the solid surface, is also affected; its calculation will be presented in another paper.

By Fourier superposition the solution for sinusoidal waves can be generalized to give the solution for other types of waves, such as are produced by cylinders of various geometries. In every case we find attenuation and diffusion of the waves. These solutions will also be published shortly.

This research was supported jointly by the United States Air Force through the Office of Scientific Research of ARDC and the Mechanics Branch of the Office of Naval Research.

REFERENCES

Alfvén, H. 1950 *Cosmical Electrodynamics*. Oxford University Press.

Cowling, T. G. 1957 *Magnetohydrodynamics*. London: Interscience.

Glauert, H. 1926 *Aerofoil and Airscrew Theory*. Cambridge University Press.

Hayes, W. D. 1949 An alternative proof of the constancy of circulation. *Quart. Appl. Math.* **7**, 235–236.

Resler, E. L., Jr. & Sears, W. R. 1958 The prospects for magneto-aerodynamics. *J. Aero. Sci.* **25**, 235–46.

Rott, N. & Cheng, H. K. 1954 Generalization of the inversion formula of thin airfoil theory. *J. Rat. Mech. Anal.* **3**, 357–82.